A Necessary and Sufficient Frequency Domain Criterion for the Passivity of SISO Sampled-Data Systems

Makan Fardad and Bassam Bamieh

***Abstract*— We present a frequency domain solution to the sampled-data passivity problem. Our analysis is exact in the sense that we take into account the intersample behavior of the system. We use frequency response (FR) operators to first obtain necessary conditions on the sampling rate T and the relative degree of the open-loop transfer function G_{11} for achieving a passive continuous-time closed-loop system. Then, assuming passivity of G_{11} and closed-loop stability, we derive a necessary and sufficient condition for discrete-time controllers that render a passive closed-loop system. We apply the obtained results to the problem of stability of haptic systems.**

***Index Terms*— Frequency response operators, haptic systems, passivity, sampled-data systems, stability.**

I. Introduction

Sampled-data systems have been the subject of extensive research for over a decade [1]–[6]. The two main approaches for the analysis of sampled-data systems have been the state-space [3] and the frequency domain [5]–[7]. We will utilize the frequency domain framework of [5], [7] where the system is represented by frequency response (FR) operators.

The importance of verifying the passivity of a control system is well understood [8]. One of its major uses is in proving closed-loop stability in the presence of uncertainty; the closed-loop interconnection of a time-invariant passive system with another time-invariant passive but otherwise uncertain system yields a stable system. An interesting and practical application of this is illustrated in [9], where sampled-data passivity is used to prove the stability of a haptic device. The authors of [9] use energy methods and physical insight to find necessary and sufficient conditions for the passivity of a sampled-data control system. It is our aim here to provide a systematic method for the analysis and synthesis of such problems, and we show that the results of [9] turn out to be a special case of the general framework presented here.

In this paper we study the passivity problem of sampled-data systems, assuming linear single-input single-output systems and controllers. We start from the passivity condition on the closed-loop system and derive a necessary and sufficient condition on the controller. The works most closely related to our's are [10], [11]. It is shown in [10] that if, in the general set up of Figure 1, the D_{11} term of G_{11} is zero (where G_{11} is the transfer function from exogenous input w to output z) then a necessary condition for the passivity of the sampled-data system is that G_{11} have relative degree one. Our work also verifies this result using different methodology. Reference [11] proposes an indirect solution to the sampled-data passivity problem in the state-space setting by checking whether the $\mathcal{H}_\infty$-norm of the Cayley transform of a sampled-data system is less than or equal to unity. Instead, in the present paper we find a direct frequency domain approach to the sampled-data passivity problem.

Our presentation is organized as follows: We introduce the FR operators used to describe a sampled-data system in the frequency domain in Section II. In Section III we first find necessary conditions on the open-loop system G_{11} and the sampling period T for achieving a passive continuous-time closed-loop system. Then, assuming passivity of the open-loop system G_{11}, we derive a necessary and sufficient condition on the discrete-time controller that renders a passive closed-loop system. The problem of the stability of haptic systems is then addressed using our passivity framework in Section IV and well-known existing results are reproduced.

Notation: Our notation is standard: We use capital letters for systems, and small letters for signals. We use the same notation for signals/systems and their Fourier transforms but the distinction will be clear from the context. P^* is the adjoint of the operator P and the complex-conjugate transpose when P is a matrix or a vector.

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II. Frequency Domain Representation of Sampled-Data Systems

In this section we describe how to find the transfer function from w to z of a closed-loop sampled-data system. Note that this is indeed a nontrivial problem because the closed-loop system is composed of both continuous-time and discrete-time subsystems, each having different frequency domain representations. The key to unifying these representations is the *lifting* operation [7].

Consider the internally stable feedback connection of the linear time-invariant continuous-time system G,

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix},$$

and the linear time-invariant discrete-time controller K, through the sample and hold devices $\mathcal{S}$ and $\mathcal{H}$ with sampling period T; see Figure 1. We assume that G_{11}, G_{12}, G_{21}, G_{22}, and K are all SISO (single-input single-output) systems.

Using the FR operator framework of [5] we arrive at

$$\begin{bmatrix} z_\theta \\ y_\theta \end{bmatrix} = \begin{bmatrix} \mathcal{G}_{11\theta} & \mathcal{G}_{12\theta} \\ \mathcal{G}_{21\theta} & \mathcal{G}_{22\theta} \end{bmatrix} \begin{bmatrix} w_\theta \\ u_\theta \end{bmatrix}.$$

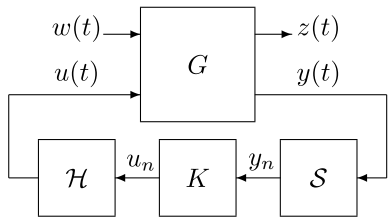

Fig. 1. The closed-loop system.

where $\mathcal{G}_{11\theta}$, $\mathcal{G}_{12\theta}$, $\mathcal{G}_{21\theta}$, $\mathcal{G}_{22\theta}$ are defined as

$$\begin{aligned}
\mathcal{G}_{11\theta} &= \begin{bmatrix} \ddots & & \\ & G_{11}(j\theta_k) & \\ & & \ddots \end{bmatrix}, \\
\mathcal{G}_{12\theta} &= \begin{bmatrix} \vdots \\ G_{12}(j\theta_k)\dfrac{1}{j\theta_k} \\ \vdots \end{bmatrix} \frac{1-e^{-j\theta}}{T}, \\
\mathcal{G}_{21\theta} &= \begin{bmatrix} \cdots & G_{21}(j\theta_k) & \cdots \end{bmatrix}, \\
\mathcal{G}_{22\theta} &= \sum_{k=-\infty}^{\infty} \frac{1-e^{-j\theta}}{j\theta_k T} G_{22}(j\theta_k), \qquad (1)
\end{aligned}$$

where

$$\theta_k := \frac{2k\pi+\theta}{T}, \quad k \in \mathbb{Z}, \quad \theta \in [0, 2\pi).$$

For a detailed derivation of the expressions for the above operators the reader is referred to [5], [12]. Let us emphasize the notational convention used above. For example, for a given value of $\theta \in [0, 2\pi)$ the diagonal operator $\mathcal{G}_{11\theta}$ is composed of equally-spaced samples at frequencies $\theta_k = (2k\pi+\theta)/T$, $k \in \mathbb{Z}$, of the transfer function $G_{11}(j\,\cdot)$, i.e., $\mathcal{G}_{11\theta} = \mathrm{diag}\{G_{11}(j\theta_k)\}_{k\in\mathbb{Z}}$.

Let $K_\theta = K(e^{j\theta})$ denote the z-transform (evaluated on the unit circle) description of the discrete-time controller. Then the closed-loop transfer function is

$$\mathcal{G}_{zw\theta} = \mathcal{G}_{11\theta} + \mathcal{G}_{12\theta}\, K_\theta\, (I - \mathcal{G}_{22\theta}\, K_\theta)^{-1}\, \mathcal{G}_{21\theta}.$$

III. Closed-Loop Passivity

This section contains the main results of the paper and uses the frequency domain framework of the previous section. We consider the passivity, from input w to output z, of the closed-loop sampled-data system in Figure 1. We first find necessary conditions on the transfer function $G_{11}(j\,\cdot)$ and the sampling period T. Then, assuming passivity of G_{11}, we derive a necessary and sufficient condition for the discrete-time controllers K that yield a passive closed-loop sampled-data system.

A. Necessary Conditions for Passivity

Consider the passivity of the internally stable closed-loop sampled-data system of Figure 1. A necessary and sufficient condition for passivity is that

$$\mathcal{G}_{zw\theta} + \mathcal{G}^*_{zw\theta} > 0 \quad \text{for all} \quad \theta \in [0, 2\pi), \qquad (2)$$

where

$$\mathcal{G}_{zw\theta} + \mathcal{G}^*_{zw\theta} > 0 \qquad \Longleftrightarrow \qquad \rho^*\left(\mathcal{G}_{zw\theta} + \mathcal{G}^*_{zw\theta}\right)\rho > 0$$

for all ρ with $\|\rho\| = 1$.

Define $L_\theta = K_\theta\,(1 - \mathcal{G}_{22\theta}\, K_\theta)^{-1}$. Then $G_{zw\theta} = \mathcal{G}_{11\theta} + \mathcal{G}_{12\theta}\, L_\theta\, \mathcal{G}_{21\theta}$, and (2) becomes

$$\begin{aligned}
&\mathcal{G}_{zw\theta} + \mathcal{G}^*_{zw\theta} = \qquad (3)\\
&\qquad \mathcal{G}_{11\theta} + \mathcal{G}^*_{11\theta} + \mathcal{G}_{12\theta}\, L_\theta\, \mathcal{G}_{21\theta} + \mathcal{G}^*_{21\theta}\, L^*_\theta\, \mathcal{G}^*_{12\theta} > 0.
\end{aligned}$$

For notational clarity we define, for a given value of θ, the bi-infinite (diagonal) matrix $\Lambda = \mathcal{G}_{11\theta}$ and the two bi-infinite vectors $\zeta = \mathcal{G}_{12\theta}$ and $\xi = (L_\theta\, \mathcal{G}_{21\theta})^*$. Equation (3) now reads

$$\Lambda + \Lambda^* + \zeta\,\xi^* + \xi\,\zeta^* > 0. \qquad (4)$$

Inequality (4) requires checking the positivity of a bi-infinite matrix. Yet an observation simplifies the problem; one can rewrite (4) as

$$\Lambda + \Lambda^* + \frac{1}{2}(\zeta+\xi)(\zeta+\xi)^* - \frac{1}{2}(\zeta-\xi)(\zeta-\xi)^* > 0,$$

or simply

$$\Lambda + \Lambda^* + \phi\,\phi^* - \psi\,\psi^* > 0, \qquad (5)$$

where $\phi := \frac{1}{\sqrt{2}}(\zeta+\xi)$ and $\psi := \frac{1}{\sqrt{2}}(\zeta-\xi)$. In (5), $\Lambda+\Lambda^*$ depends on G_{11} only, while the two *one-dimensional* operators (i.e., operators with one-dimensional range space) $\phi\,\phi^*$ and $\psi\,\psi^*$ contain the controller K. In other words, in the frequency domain, closing the loop with a discrete controller K_θ constitutes a two-dimensional perturbation of the infinite-dimensional open-loop system $\mathcal{G}_{11\theta}$.

Lemma 1: A necessary condition for closed-loop passivity is that $\mathcal{G}_{11\theta} + \mathcal{G}^*_{11\theta}$ have at most one nonpositive eigenvalue (equivalently, nonpositive diagonal element) for every $\theta \in [0, 2\pi)$.

Proof: Assume $D_{11} = 0$, i.e., G_{11} has no direct feedthrough term. Then for every θ the operator $\mathcal{G}_{11\theta}$ is compact [13, Chap. 8]. For a given θ consider the compact operator $\Lambda+\Lambda^* = \mathcal{G}_{11\theta}+\mathcal{G}^*_{11\theta}$. Let us denote the ordered real and nonpositive eigenvalues of a self-adjoint diagonal operator P by $\lambda_i(P)$, $i = 1, 2, \cdots$; $\lambda_1(P) \le \lambda_2(P) \le \cdots \le 0$. Assume that $\Lambda+\Lambda^*$ has two or more nonpositive eigenvalues, $\lambda_1(\Lambda+\Lambda^*) \le \lambda_2(\Lambda+\Lambda^*) \le 0$. Then from [14], [15] it follows that

$$\lambda_i(\Lambda+\Lambda^* - \psi\,\psi^*) \le \lambda_i(\Lambda+\Lambda^*), \qquad i = 1, 2, \cdots,$$

which means that $\Lambda + \Lambda^* - \psi\,\psi^*$ will also have at least two nonpositive eigenvalues. Now using the fact that the eigenvalues of $\Lambda + \Lambda^* - \psi\,\psi^*$ and $\Lambda + \Lambda^* + \phi\,\phi^* - \psi\,\psi^*$ interlace [14], [15] we have that

$$\lambda_1(\Lambda+\Lambda^*+\phi\phi^*-\psi\psi^*) \le \lambda_2(\Lambda+\Lambda^*-\psi\psi^*) \le 0.$$

Hence $\Lambda+\Lambda^*+\phi\phi^*-\psi\psi^*$ is not positive and the closed-loop system is not passive.

In the case of $D_{11} \ne 0$ the spectra of the operators $\Lambda+\Lambda^*$, $\Lambda+\Lambda^*-\psi\,\psi^*$, and $\Lambda+\Lambda^*+\phi\,\phi^*-\psi\,\psi^*$ will all be shifted by the amount $D_{11}+D^*_{11}$. Thus the results obtained above extend to the case of $D_{11} \ne 0$ and the proof is complete. ■

Recall that since $\mathcal{G}_{11\theta} + \mathcal{G}_{11\theta}^* = \mathrm{diag}\{(G_{11} + G_{11}^*)(j\theta_k)\}_{k\in\mathbb{Z}}$, the eigenvalues of $\mathcal{G}_{11\theta} + \mathcal{G}_{11\theta}^*$ are nothing but the set of real numbers $\{(G_{11} + G_{11}^*)(j\theta_k)\}_{k\in\mathbb{Z}}$. Thus by Lemma 1 if for some $\theta \in [0, 2\pi)$ we have

$$(G_{11} + G_{11}^*)(j\theta_k) \;\le\; 0 \quad \text{for more than one} \quad k \in \mathbb{Z},$$

then $\mathcal{G}_{11\theta} + \mathcal{G}_{11\theta}^*$ will have more than one nonpositive eigenvalue and the closed-loop system can not be passive.

Theorem 2: For a given sampling period T, a necessary condition for closed-loop passivity is that the Nyquist plot of G_{11} not reside in the left-half of the complex plane for any frequency interval of length greater than or equal to $2\pi/T$.

Proof: It is easy to see that if the condition in the theorem statement does not hold then $(G_{11} + G_{11}^*)(j\theta_k)$ will be nonpositive for at least two consecutive integers $\bar{k}$ and $\bar{k}+1$, and thus $\mathcal{G}_{11\theta} + \mathcal{G}_{11\theta}^*$ will have at least two nonpositive eigenvalues. The claim now follows from Lemma 1. ■

Theorem 2 leads to the following necessary conditions for closed-loop passivity.

(a) For the closed-loop system to be passive we need $D_{11} + D_{11}^* \ge 0$, where D_{11} is the direct feed-through term of G_{11}. To see this, note that the Nyquist plot of G_{11} tends to D_{11} as ω tends to $\pm\infty$. Therefore, if $D_{11} + D_{11}^* < 0$ the inequality $(G_{11} + G_{11}^*)(j\omega) < 0$ will hold on an unbounded frequency interval $\omega \in [\omega_0, \infty)$ for some ω_0, and closed-loop passivity can not be achieved by Theorem 2. Furthermore, the assumption $D_{11} = 0$ places a restriction on the relative degree of $G_{11}(s)$. For example, if $D_{11} = 0$ then $G_{11}(s)$ can not have relative degree two, otherwise as ω tends to $\pm\infty$ the Nyquist plot of G_{11} will approach the origin with a ± 180 degree phase. Thus the inequality $(G_{11} + G_{11}^*)(j\omega) \le 0$ will hold on an unbounded frequency interval and closed-loop passivity can not be achieved.

(b) If the system G is such that $(G_{11} + G_{11}^*)(j\omega) \le 0$ for $\omega \in [\omega_1, \omega_2]$, or equivalently the Nyquist plot G_{11} resides in the left-half of the complex plane for $\omega \in [\omega_1, \omega_2]$, then from Theorem 2 the sampling period T should be chosen such that

$$\frac{2\pi}{T} \;>\; |\omega_2 - \omega_1| \quad \Longrightarrow \quad T \;<\; \frac{2\pi}{|\omega_2 - \omega_1|}.$$

B. Necessary and Sufficient Conditions for Passivity Assuming Passive G_{11}

From Lemma 1 we know that a necessary condition for closed-loop passivity is that the operator $\mathcal{G}_{11\theta} + \mathcal{G}_{11\theta}^*$ have at most one nonpositive eigenvalue, where $\mathcal{G}_{11\theta} + \mathcal{G}_{11\theta}^* = \mathrm{diag}\{(G_{11} + G_{11}^*)(j\theta_k)\}_{k\in\mathbb{Z}}$. This motivates us to assume passivity of the open-loop system G_{11}, i.e., we assume that G_{11} is stable and

$$(G_{11} + G_{11}^*)(j\omega) \;>\; 0 \quad \text{for all} \quad \omega \in \mathbb{R}.$$

This means that $\Lambda + \Lambda^* = \mathcal{G}_{11\theta} + \mathcal{G}_{11\theta}^* > 0$, and thus $\Lambda + \Lambda^* + \phi\,\phi^* > 0$. Applying the Schur complement we get the following set of equivalent inequalities

$$\begin{gathered} \Lambda + \Lambda^* + \phi\,\phi^* - \psi\,\psi^* \;>\; 0 \\ \Updownarrow \\ \begin{bmatrix} \Lambda + \Lambda^* + \phi\,\phi^* & \psi \\ \psi^* & 1 \end{bmatrix} \;>\; 0 \\ \Updownarrow \\ 1 - \psi^* (\Lambda + \Lambda^* + \phi\,\phi^*)^{-1}\,\psi \;>\; 0. \end{gathered} \tag{6}$$

We have effectively transformed the question of the positivity of a bi-infinite matrix to that of a scalar. This, however, comes at a price; the inverse of the bi-infinite matrix $\Lambda + \Lambda^* + \phi\,\phi^*$ has to be found. But for this we can use the matrix inversion lemma

$$\begin{aligned} (\Lambda + \Lambda^* + \phi\,\phi^*)^{-1} \;=\; & (\Lambda + \Lambda^*)^{-1} - (\Lambda + \Lambda^*)^{-1} \\ & \times \phi\left(1 + \phi^* (\Lambda + \Lambda^*)^{-1}\,\phi\right)^{-1} \phi^* (\Lambda + \Lambda^*)^{-1}, \end{aligned}$$

to get

$$\begin{aligned} \psi^* (\Lambda + \Lambda^* + \phi\,\phi^*)^{-1}\,\psi \;=\; & \psi^* (\Lambda + \Lambda^*)^{-1}\,\psi \\ & - \frac{\left(\psi^* (\Lambda + \Lambda^*)^{-1}\,\phi\right)\left(\phi^* (\Lambda + \Lambda^*)^{-1}\,\psi\right)}{1 + \phi^* (\Lambda + \Lambda^*)^{-1}\,\phi}. \end{aligned}$$

Thus inequality (6) becomes

$$1 + \frac{|\psi^* (\Lambda + \Lambda^*)^{-1}\,\phi|^2}{1 + \phi^* (\Lambda + \Lambda^*)^{-1}\,\phi} \;>\; \psi^* (\Lambda + \Lambda^*)^{-1}\,\psi. \tag{7}$$

Since $\Lambda + \Lambda^* > 0$ we can define a new inner product on a subspace of ℓ^2,

$$\langle \rho, \eta \rangle_\Lambda \;:=\; \rho^* (\Lambda + \Lambda^*)^{-1}\,\eta,$$

which induces a new norm $\|\eta\|_\Lambda^2 := \langle \eta, \eta \rangle_\Lambda$. Substituting $\phi = (\zeta + \xi)/\sqrt{2}$ and $\psi = (\zeta - \xi)/\sqrt{2}$ back into (7) and simplifying we arrive at

$$|1 + \langle \zeta, \xi \rangle_\Lambda|^2 \;>\; \|\zeta\|_\Lambda^2\, \|\xi\|_\Lambda^2,$$

or equivalently

$$|1 + \langle \mathcal{G}_{12\theta}, \mathcal{G}_{21\theta}^*\, L_\theta^* \rangle_\Lambda|^2 \;>\; \|\mathcal{G}_{12\theta}\|_\Lambda^2\, \|\mathcal{G}_{21\theta}^*\, L_\theta^*\|_\Lambda^2. \tag{8}$$

Replacing L_θ with $K_\theta\,(1 - \mathcal{G}_{22\theta}\,K_\theta)^{-1}$, (8) simplifies to

$$|K_\theta^{-1} - (\mathcal{G}_{22\theta} - \langle \mathcal{G}_{12\theta}, \mathcal{G}_{21\theta}^* \rangle_\Lambda)|^2 \;>\; \|\mathcal{G}_{12\theta}\|_\Lambda^2\, \|\mathcal{G}_{21\theta}^*\|_\Lambda^2. \tag{9}$$

We have thus proved the following theorem, which is the main result of this section.

Theorem 3: Assuming passivity of the open-loop system G_{11}, an internally stable closed-loop sampled-data system is passive if and only if K_θ^{-1} lies outside a disk with center

$$\mathcal{G}_{22\theta} - \langle \mathcal{G}_{12\theta}, \mathcal{G}_{21\theta}^* \rangle_\Lambda$$

and radius

$$\|\mathcal{G}_{12\theta}\|_\Lambda\, \|\mathcal{G}_{21\theta}^*\|_\Lambda$$

for all $\theta \in [0, 2\pi)$, where $\langle\,\cdot\,,\,\cdot\,\rangle_\Lambda$ is a weighted inner product with weighting $(\mathcal{G}_{11\theta} + \mathcal{G}_{11\theta}^*)^{-1}$ and $\|\cdot\|_\Lambda^2$ is the corresponding weighted norm. ■

Remark 1: If D_{11} is zero, i.e., G_{11} has no direct feedthrough term, then $\mathcal{G}_{11\theta} + \mathcal{G}_{11\theta}^*$ is a compact operator [13, Chap. 8] and hence $(\mathcal{G}_{11\theta} + \mathcal{G}_{11\theta}^*)^{-1}$ will be an unbounded

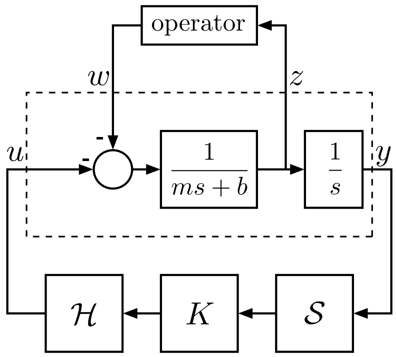

Fig. 2. Model of a haptic system considered in [9].

operator. This puts certain constraints on the admissible ϕ and ψ so that each term in (7) is bounded, which translates to restrictions on the relative degrees of $G_{12}(s)$ and $G_{21}(s)$ so that each term in (9) remains finite. For example, it can be shown that if $G_{11}(s)$ has relative degree one then $G_{12}(s)$ and $G_{21}(s)$ should have relative degrees one and two, respectively. ∎

Remark 2: Notice that the evaluation of the terms $\langle \cdot\,,\,\cdot \rangle_\Lambda$ and $\|\cdot\|_\Lambda^2$ involves the computation of infinite sums. Hagiwara *et al* [5] give a method for the calculation of $\sum_{k=-\infty}^{\infty} G(j\theta_k)$ using the impulse modulation formula. Namely, for a system G with state-space description (A, B, C),

$$\sum_{k=-\infty}^{\infty} G(j\theta_k) \;=\; e^{j\theta}\, C\, (e^{j\theta} \,-\, e^{AT})^{-1}\, B.$$

IV. Example: Passivity of Haptic Systems

In this section we apply the results of the previous section to find derive conditions for the passivity of a haptic system [9]. A haptic system is a device that mechanically simulates a virtual object or environment. One application of haptic systems is performing remote surgical operations. In Figure 2 the human operator is shown by the “operator” block and we refer to what the operator sees, i.e., the system from w to z, as the haptic system. The haptic system is composed of an actuator handle which the human operator holds (shown by the $\frac{1}{ms+b}$ block in the figure) and the virtual environment simulated by the controller K, which interacts with the actuator through sample and hold devices. Here z and y are the velocity and position of the haptic system, respectively, and u is the force feedback.

The human operator is considered to behave as a *passive but otherwise arbitrary* impedance [9]. Therefore, since the feedback connection of passive systems is stable [8] the stability of the overall haptic+human system is guaranteed if the haptic system from w to z is passive. If we consider the dashed box as the system G, we can apply the results of the previous section to find conditions on K that yield the passivity of G_{zw}.

We have

$$G_{11}(s) \;=\; -\frac{1}{ms+b}, \qquad G_{12}(s) \;=\; -\frac{1}{ms+b},$$
$$G_{21}(s) \;=\; -\frac{1}{s(ms+b)}, \quad G_{22}(s) \;=\; -\frac{1}{s(ms+b)},$$

where $b > 0$ is the damping coefficient and $m > 0$ is the mass of the haptic actuator handle. It is easy to see that the inequality $\big(G_{11} + G_{11}^*\big)(j\omega) > 0$ holds for all $\omega \in \mathbb{R}$.

Simple calculations give

$$\mathcal{G}_{22\theta} \,-\, \langle \mathcal{G}_{12\theta}, \mathcal{G}_{21\theta}^* \rangle_\Lambda \;=\; \frac{1 \,-\, e^{-j\theta}}{2\,b}\, \frac{T}{4\,(\sin\theta/2)^2},$$
$$\|\mathcal{G}_{12\theta}\|_\Lambda^2\, \|\mathcal{G}_{21\theta}^*\|_\Lambda^2 \;=\; \frac{|1 \,-\, e^{-j\theta}|^2}{b^2}\, \frac{T^2}{64\,(\sin\theta/2)^2}.$$

Defining

$$r_\theta \;:=\; -\,(1 \,-\, e^{-j\theta})\, \frac{T}{4\,(\sin\theta/2)^2},$$

substituting into (9), and simplifying, we arrive at

$$\left| \frac{r_\theta\, K_\theta}{2\,b \,+\, r_\theta\, K_\theta} \right| \;<\; 1 \tag{10}$$

which has to be satisfied for every $\theta \in [0,\, 2\pi)$. This matches precisely the passivity condition given in [9], which was obtained using energy methods and physical insight. We note that at this point the inequality (10) is a necessary condition for passivity. It will also become a sufficient condition once it is established that the discrete-time controller K achieves closed-loop stability.

V. Conclusions and Future Work

We use the frequency domain and frequency response (FR) operators to investigate the passivity of sampled-data systems. We show that, with regards to passivity, closing the loop with a discrete controller K constitutes a two-dimensional perturbation of the infinite-dimensional FR operator corresponding to the open-loop system G_{11}. Using this we derive conditions on G_{11} (the transfer function from exogenous input to output of the open-loop system) and an upper bound on the sampling period T that are necessary for closed-loop passivity of the sampled-data system.

We then assume passivity of G_{11} and closed-loop stability to derive a necessary and sufficient frequency domain condition on the discrete controller that guarantees closed-loop passivity of the sampled-data system. Namely, we show that the inverse of the controller transfer function, K_θ^{-1}, should reside in the exterior of a disk in the complex plane described by the system parameters, for all frequencies $\theta \in [0, 2\pi)$.

Future work in this direction would include expressing the necessary and sufficient passivity conditions in state-space, and relaxing the passivity condition on G_{11}.

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